Przykładowy zestaw zaliczeniowy

1. Obliczyć pochodne cząstkowe rzędu 1 i 2 funkcji:

$$f(x,y) = (y\sin y)^{\operatorname{arctg} x}$$

Odp.:
$$\frac{\partial f}{\partial x} = (y\sin y)^{\operatorname{arctg} x} \ln(y\sin y)\cdot\frac{1}{x^2+1}$$

$$\frac{\partial f}{\partial y} = \operatorname{arctg} x\,(y\sin y)^{\operatorname{arctg} x - 1}\cdot(\sin y + y\cos y)$$

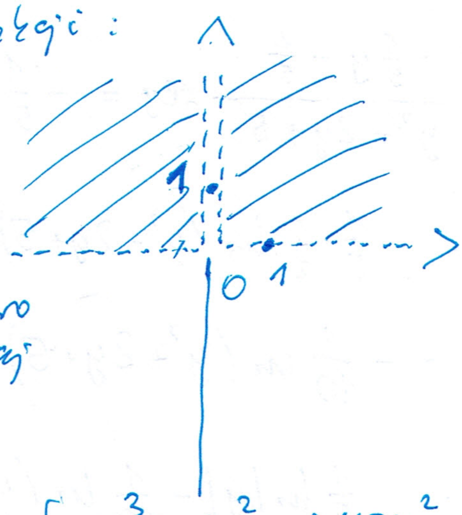
$$\frac{\partial^2 f}{\partial x^2} = (y\sin y)^{\operatorname{arctg} x}\big(\ln(y\sin y)\big)^2\cdot\frac{1}{(x^2+1)^2} + (y\sin y)^{\operatorname{arctg} x}\ln(y\sin y)\cdot\frac{-2x}{(x^2+1)^2}$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x} = \frac{1}{x^2+1}(y\sin y)^{\operatorname{arctg} x - 1}\cdot(\sin y + y\cos y) +$$
$$+ \operatorname{arctg} x\,(y\sin y)^{\operatorname{arctg} x - 1}\ln(y\sin y)\cdot\frac{1}{x^2+1}\cdot(\sin y + y\cos y)$$

$$\frac{\partial^2 f}{\partial y^2} = \operatorname{arctg} x\cdot(\operatorname{arctg} x - 1)(y\sin y)^{\operatorname{arctg} x - 2}(\sin y + y\cos y)^2 +$$
$$+ \operatorname{arctg} x\,(y\sin y)^{\operatorname{arctg} x - 1}\cdot(\cos y + \cos y - y\sin y)$$

2. Wyznaczyć ekstrema lokalne funkcji:

$$f(x,y) = \frac{1}{x} + \frac{x}{y} + \ln y$$



I Dziedzina: $x \neq 0,\ y \neq 0,\ y > 0$.

obrazowo tutaj

$$\begin{cases} \frac{\partial f}{\partial x} = -\frac{1}{x^2} + \frac{1}{y} = 0 \\ \frac{\partial f}{\partial y} = -\frac{x}{y^2} + \frac{1}{y} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} -\frac{1}{x^2} + \frac{x}{y^2} = 0 \\ -\frac{1}{x^2} + \frac{1}{y} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{x}{y^2} = \frac{1}{x^2} \\ \frac{1}{y} = \frac{1}{x^2} \end{cases} \Leftrightarrow \begin{cases} x^3 = y^2 \\ x^2 = y \end{cases} \Leftrightarrow \begin{cases} y = x^2 \\ x^3 = x^4 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = x^2 \\ x^3(1-x) = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \vee x = 1 \\ \text{sprz.} \\ y = x^2 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ y = 1 \end{cases}$$

Punkt „podejrzany" to $A = (1,1)$.

Obliczamy

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(-x^{-2}\right) = 2x^{-3}$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{1}{y^2} \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(-xy^{-2} + \frac{1}{y}\right) = 2xy^{-3} - \frac{1}{y^2}$$

i hesjan w A:

$$H_A f = \det \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = 3 > 0 \Rightarrow$$ jest ekstremum i jest to minimum lokalne.

3. Rozwiązać równania różniczkowe:

a) $y'\sqrt{4-x^2} = \left(y^2 + 2y + 5\right) y$

Rozw. $$\int \frac{dy}{y(y^2+2y+5)} = \int \frac{dx}{\sqrt{4-x^2}}$$

$$\frac{1}{y(y^2+2y+5)} = \frac{A}{y} + \frac{By+C}{y^2+2y+5}$$

$$1 = A(y^2+2y+5) + By^2 + Cy = y^2(A+B) + y(2A+C) + 5A$$

$$\begin{cases} A+B=0 \\ 2A+C=0 \\ 5A=1 \end{cases} \qquad \begin{cases} A = \frac{1}{5} \\ B = -\frac{1}{5} \\ C = -\frac{2}{5} \end{cases}$$

$$\int \frac{-\frac{1}{5}y - \frac{2}{5}}{y^2+2y+5}\,dy = -\frac{1}{5}\int \frac{y+2}{y^2+2y+5}\,dy = -\frac{1}{5}\int \frac{(2y+2)\cdot\frac{1}{2} + 1}{y^2+2y+5}\,dy =$$

$$= -\frac{1}{5}\cdot\frac{1}{2}\ln\left(y^2+2y+5\right) - \frac{1}{5}\int \frac{dy}{(y+1)^2+2^2} =$$

$$= -\frac{1}{10}\ln\left(y^2+2y+5\right) - \frac{1}{5}\cdot\frac{1}{2}\operatorname{arctg}\frac{y+1}{2} + C$$

Odp. $$\frac{1}{5}\ln|y| - \frac{1}{10}\ln\left(y^2+2y+5\right) - \frac{1}{10}\operatorname{arctg}\frac{y+1}{2} = \arcsin\frac{x}{2} + C$$

b) $xyy' + x^2 + y^2 = 0$

(jest to równanie jednorodne względem x i y) więc obustronnie podzielimy przez x^2

$$\frac{y}{x} y' + 1 + \left(\frac{y}{x}\right)^2 = 0$$

$u = \frac{y}{x}$, $y = ux \Rightarrow y' = u'x + u$ i podstawiamy:

$$u(u'x+u) + 1 + u^2 = 0$$

$$u(u'x+u) = -u^2 - 1$$

$$u'x + u = \frac{-u^2-1}{u}$$ (lub $u=0$, czyli $y=0$, ale $y=0$ nie spełnia początkowego równania)

$$u'x = \frac{-u^2-1}{u} - u$$

$$u'x = -u - \frac{1}{u} - u = -2u - \frac{1}{u} = -\frac{2u^2+1}{u}$$

$$\frac{du}{dx} \cdot x = -\frac{2u^2+1}{u}$$

$$\frac{1}{4}\int \frac{4u\,du}{2u^2+1} = -\int \frac{dx}{x}$$

dopisujemy $\left\{ \begin{array}{l} t = 2u^2+1 \\ dt = 4u\,du \end{array} \right\} = \frac{1}{4}\int \frac{dt}{t} = \frac{1}{4}\ln|t| (+C) = \frac{1}{4}\ln(\underbrace{2u^2+1}_{>0}) (+C)$

Dalej: $\frac{1}{4}\ln(2u^2+1) = -\ln|x| + C$

i wracamy do x, y: $\frac{1}{4}\ln\left(2\left(\frac{y}{x}\right)^2 + 1\right) = -\ln|x| + C$

i to wystarczy.

c) $y' - y\,tg\,x = \cos x$, spełniające warunek $y(0)=1$

I Rozwiązujemy ogólnie wg wzoru:

$$y = C(x)e^{-\int p}$$

$$C(x) = \int q e^{\int p}$$

$$p(x) = -tg\,x \quad , \quad q(x) = \cos x$$

$$\int p = -\int \frac{\sin x}{\cos x}dx = \int \frac{-\sin x}{\cos x}dx = \ln|\cos x| \; (+C)$$

$$e^{\int p} = e^{\ln|\cos x|} = |\cos x| . \qquad \left\{ \begin{matrix} \text{np. } t = \cos x \\ dt = -\sin x\,dx \end{matrix} \right\} \qquad e^{-\int p} = \frac{1}{e^{\int p}} = \frac{1}{|\cos x|}$$

$$C(x) = \int q e^{\int p} = \int \cos x\, e^{\ln|\cos x|} dx = \pm \int \cos^2 x\,dx =$$

$$(|\cos x| = \pm \cos x)$$

$$= \pm \int \frac{1+\cos 2x}{2}dx = \pm\left(\frac{1}{2}x + \frac{1}{4}\sin 2x \pm C\right)$$

Zatem

$$y = \pm\left(\frac{1}{2}x + \frac{1}{4}\sin 2x + C\right)\cdot\frac{1}{\pm\cos x}$$

$$y = \left(\frac{1}{2}x + \frac{1}{4}\sin 2x + C\right)\cdot\frac{1}{\cos x}$$

II Uwzględniamy warunek początkowy

$$1 = y(0) = C\cdot\frac{1}{\cos 0} = C .$$

Ostatecznie, $y = \left(\frac{1}{2}x + \frac{1}{4}\sin 2x + 1\right)\cdot\frac{1}{\cos x}$

d) $y'' + 2y' + 5y = 2\cos 3x$

$$r^2 + 2r + 5 = 0$$

$$\Delta = -16 = (4i)^2$$

$$r_{1,2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$$y_1 = e^{-x}\cos 2x, \quad y_2 = e^{-x}\sin 2x$$

Przewidujemy $y_s = a\cos 3x + b\sin 3x$

Odp. $y = -\frac{2}{13}\cos 3x - \frac{3}{13}\sin 3x + C_1 e^{-x}\cos 2x + C_2 e^{-x}\sin 2x$, $C_1, C_2 \in \mathbb{R}$.